

LOGARITHMIC CONVEXITY OF FIXED POINTS OF STOCHASTIC KERNEL OPERATORS

ALJOŠA PEPERKO

ABSTRACT. In this paper we prove results on logarithmic convexity of fixed points of stochastic kernel operators. These results are expected to play a key role in the economic application to strategic market games.

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1. INTRODUCTION

In this paper we prove results on logarithmic convexity of fixed points of stochastic kernel operators. These result generalize and extend the finite dimensional results from [25]. We extend the results of [25] even in the case of $n \times n$ matrices. These results were originally motivated by economic considerations with application to strategic market games (see e.g. [27], [6], [26] and the references cited there). More precisely, they were, together with Kakutani's fixed point theorem (see e.g. [17]), a crucial step in the proof of the existence of Nash equilibria in the model of an exchange economy with complete markets studied in [27]. Thus our results are expected to be a key ingredient in the development of the infinite dimensional generalization of this economic model.

The paper is organized as follows. In the rest of the current section we recall basic definitions and facts, which we will need in our proofs and we prove our main results in Section 2 (Theorem 2.4). In Section 3 we apply our results to finite or infinite non-negative matrices that define weighted operators on sequence spaces (Corollary 3.1) and explain how our results fit into the economic setting of strategic market games (Remark 3.2(ii)). In Remarks 3.2(i) and (iii) we also point out possible applications to Arrow-Debreu model and to open Leontief model of an economy and to Google Page-rank model of internet usage. For these particular applications it might suffice, if we restricted our results to the

setting of Section 3, i.e., to (infinite dimensional) non-negative matrices. However, it is well-known that kernel (integral) operators play a very important, often even central, role in a variety of applications from differential and integro-differential equations, problems from physics (in particular from thermodynamics), engineering, statistical and economic models, etc. (see e.g. [16], [24], [7], [18], [11], [21], [28], [20] and the references cited there). Therefore we choose to present our results in this more general setting.

Let μ be a σ -finite positive measure on a σ -algebra \mathcal{M} of subsets of a non-void set X . Let $M(X, \mu)_+$ be the cone of all equivalence classes of (almost everywhere equal) μ -measurable functions on X whose values lie in $[0, \infty]$. For $f \in M(X, \mu)_+$ we write $f > 0$ if and only if $\mu\{x \in X : f(x) > 0\} > 0$, and $f \gg 0$ if and only if $f(x) > 0$ for almost all $x \in X$.

Let $M_0(X, \mu)$ be the vector lattice of all equivalence classes of (almost everywhere equal) complex μ -measurable functions on X . A vector subspace $L \subseteq M_0(X, \mu)$ is called an *ideal* if $f \in M_0(X, \mu)$, $g \in L$ and $|f| \leq |g|$ a.e. imply that $f \in L$. A subset $C \subset L$ is called a *wedge*, if $f + g \in C$ and $\alpha f \in C$ for all $f, g \in C$ and $\alpha > 0$. For an ideal $L \subseteq M_0(X, \mu)$ we say that X is the *carrier* of L if there is no subset Y of X of strictly positive measure with the property that $f = 0$ a.e. on Y for all $f \in L$ (see [29]).

A seminorm $\|\cdot\|$ on the ideal $L \subseteq M_0(X, \mu)$ is called a *lattice seminorm* (also Riesz seminorm) if $f \in M_0(X, \mu)$, $g \in L$ and $|f| \leq |g|$ a.e. imply that $\|f\| \leq \|g\|$. A *lattice norm* is a lattice seminorm which is also a norm. Recall that an ideal $L \subseteq M_0(X, \mu)$ equipped with a lattice norm $\|\cdot\|$ is sometimes called a *normed Köthe space* ([29, p. 421]) and that a complete normed Köthe space is called a *Banach function space*. Standard examples of Banach function spaces are Euclidean spaces, the space c_0 of all null convergent sequences (equipped with the usual norms and the counting measure), the well known $L_p(X, \mu)$ spaces ($1 \leq p \leq \infty$) and other less known examples such as Orlicz, Lorentz, Marcinkiewicz and more general rearrangement-invariant spaces (see e.g. [9], [10] and the references cited there), which are important e.g. in interpolation theory. The set of all normed Köthe spaces (or of all Banach function spaces) is also closed under all cartesian products $L = E \times F$ equipped with the norm $\|(x, y)\|_L = \max\{\|x\|_E, \|y\|_F\}$.

The cone of non-negative elements in L is denoted by L_+ . By an *operator* from an ideal L to an ideal N we always mean a linear operator from L to N . An operator $T : L \rightarrow N$ is said to be *positive* if $Tf \in N_+$ for all $f \in L_+$. Given operators S and T from L to N , we write $T \leq S$ if the operator $S - T$ is positive. A positive operator $T : L \rightarrow N$ is called

strictly positive if $Tf \gg 0$ whenever $f \gg 0$, $f \in L_+$. Recall that a positive operator between a Köthe space L and a Banach function space N is always bounded (see [1], [3]), i.e., its operator norm is finite:

$$\|A\| = \sup\{\|Af\|_N : f \in L, \|f\|_L \leq 1\} = \sup\{\|Af\|_N : f \in L_+, \|f\|_L \leq 1\} < \infty.$$

Let (X, μ) and (Y, ν) be σ -finite measure spaces, L and N ideals in $M_0(Y, \nu)$ and $M_0(X, \mu)$ respectively, such that Y is the carrier of L . An operator $K : L \rightarrow N$ is called a *kernel operator* if there exists a $\mu \times \nu$ -measurable function $k(x, y)$ on $X \times Y$ such that, for all $f \in L$ and for almost all $x \in X$, $\int_X |k(x, y)f(y)| d\nu(y) < \infty$ and $(Kf)(x) = \int_X k(x, y)f(y) d\nu(y)$. One can check that a kernel operator K is positive iff its kernel k is non-negative almost everywhere (see [29]).

For the theory of normed Köthe spaces, Banach function spaces, Banach lattices, cones, wedges, positive operators and applications e.g. in financial mathematics we refer the reader to the books [1], [5], [29], [3], [2], [9] and the references cited there.

Let $L \subset M_0(X, \mu)$ be an ideal. A positive kernel operator S on L with kernel $s(x, y)$ is called substochastic, if $s(y) = \int_X s(x, y) dx \leq 1$ for almost all $y \in X$. The operator S is called stochastic, if $s(y) = 1$ for almost all $y \in X$, and S is called strictly substochastic if $s(y) < 1$ for almost all $y \in X$. Note that if S is a stochastic operator on $L^1(X, \mu)$, then S is a Markov operator (see e.g. [24]).

2. MAIN RESULTS

Firstly, let us recall a version of a result from [23, Corollary 3.2], which follows from the sharpened version of Young's inequality

$$(1) \quad x^\alpha y^{1-\alpha} = \inf_{t>0} \left\{ \alpha t^{\frac{1}{\alpha}} x + (1-\alpha)t^{-\frac{1}{1-\alpha}} y \right\},$$

where $x, y \geq 0$ and $\alpha \in (0, 1)$.

Proposition 2.1. *Let $L \subset M_0(X, \mu)$ be an ideal and $\alpha_i > 0$, $i = 1, \dots, m$, such that $\sum_{i=1}^m \alpha_i = 1$. Then $f_1^{\alpha_1} f_2^{\alpha_2} \dots f_m^{\alpha_m} \in L$ and*

$$(2) \quad \|f_1^{\alpha_1} f_2^{\alpha_2} \dots f_m^{\alpha_m}\|_L \leq \|f_1\|_L^{\alpha_1} \|f_2\|_L^{\alpha_2} \dots \|f_m\|_L^{\alpha_m}$$

for all $f_i \in L$ and any lattice seminorm $\|\cdot\|_L$.

In particular, for $\alpha_i > 0$, $i = 1, \dots, m$, such that $\sum_{i=1}^m \alpha_i = 1$, the following well-known generalized Hölder's inequality holds:

$$(3) \quad \int f_1^{\alpha_1} f_2^{\alpha_2} \dots f_m^{\alpha_m} d\mu \leq \left(\int f_1 d\mu \right)^{\alpha_1} \left(\int f_2 d\mu \right)^{\alpha_2} \dots \left(\int f_m d\mu \right)^{\alpha_m},$$

where f_1, \dots, f_m are non-negative measurable functions on X .

The following result follows easily.

Proposition 2.2. *Let L and N be ideals in $M_0(Y, \nu)$ and $M_0(X, \mu)$ respectively, such that Y is the carrier of L . If $K : L \rightarrow N$ is a positive kernel operator and $\alpha_i > 0$, $i = 1, \dots, m$, such that $\sum_{i=1}^m \alpha_i = 1$, then*

$$(4) \quad K(f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m}) \leq (Kf_1)^{\alpha_1} (Kf_2)^{\alpha_2} \cdots (Kf_m)^{\alpha_m}$$

for all $f_1, \dots, f_m \in L_+$.

If, in addition, $\|\cdot\|_N$ is a lattice seminorm on N , then

$$(5) \quad \|K(f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m})\|_N \leq \|(Kf_1)^{\alpha_1} (Kf_2)^{\alpha_2} \cdots (Kf_m)^{\alpha_m}\|_N \leq \|Kf_1\|_N^{\alpha_1} \|Kf_2\|_N^{\alpha_2} \cdots \|Kf_m\|_N^{\alpha_m}.$$

Proof. For almost all $x \in X$ we have by (3)

$$\begin{aligned} K(f_1^{\alpha_1} \cdots f_m^{\alpha_m})(x) &= \int_X k(x, y) f_1^{\alpha_1}(y) \cdots f_m^{\alpha_m}(y) d\nu(y) \\ &= \int_X (k(x, y) f_1(y))^{\alpha_1} \cdots (k(x, y) f_m(y))^{\alpha_m} d\nu(y) \leq (Kf_1(x))^{\alpha_1} \cdots (Kf_m(x))^{\alpha_m}, \end{aligned}$$

which proves (4).

The inequalities (5) now follow from (4) and (2). \square

Let $0 < \alpha < 1$ and $f_i, g_i \in M(X, \mu)_+$ for $i = 1, \dots, m$. As already noted (but applied in a different way) in [23, Inequality (15)] an application of Hölder's inequality or (1) gives

$$(6) \quad f_1^\alpha g_1^{1-\alpha} + \cdots + f_m^\alpha g_m^{1-\alpha} \leq (f_1 + \cdots + f_m)^\alpha (g_1 + \cdots + g_m)^{1-\alpha}.$$

This implies the following result.

Proposition 2.3. *Let L and N be ideals in $M_0(Y, \nu)$ and $M_0(X, \mu)$ respectively, such that Y is the carrier of L . If $K : L \rightarrow N$ is a positive kernel operator and $\alpha_i > 0$, $i = 1, \dots, m$, such that $\sum_{i=1}^m \alpha_i = 1$, then*

$$(7) \quad \begin{aligned} K(f_1^\alpha g_1^{1-\alpha} + \cdots + f_m^\alpha g_m^{1-\alpha}) &\leq K((f_1 + \cdots + f_m)^\alpha (g_1 + \cdots + g_m)^{1-\alpha}) \\ &\leq (K(f_1 + \cdots + f_m))^\alpha (K(g_1 + \cdots + g_m))^{1-\alpha}. \end{aligned}$$

for all $f_i, g_i \in L_+$, $i = 1, \dots, m$.

If, in addition, $\|\cdot\|_N$ is a lattice seminorm on N , then

$$\|K(f_1^\alpha g_1^{1-\alpha} + \cdots + f_m^\alpha g_m^{1-\alpha})\|_N \leq \|K((f_1 + \cdots + f_m)^\alpha (g_1 + \cdots + g_m)^{1-\alpha})\|_N$$

$$(8) \quad \leq \|K(f_1 + \dots + f_m)\|_N^\alpha \|K(g_1 + \dots + g_m)\|_N^{1-\alpha}.$$

Proof. The inequalities (7) follow from (6) and (4), while the inequalities (8) follow from (7) and (5). \square

Now we prove our main result which is an infinite dimensional generalization and extension of the main result from [25], which was a crucial step in the proof of the existence of Nash equilibria in the model of an exchange economy with complete markets studied in [27]. For a substochastic kernel operator S on L we define

$$C(S) = \{f \in L_+ : f \gg 0 \text{ and there exists a stochastic } A \geq S \text{ such that } Af = f\}.$$

The meaning of the set $C(S)$ in the economic setting of strategic market games is explained in Remark 3.2(ii).

Theorem 2.4. *Let $L \subset L^1(X, \mu)$ be an ideal, such that X is the carrier of L . Assume that S a substochastic kernel operator on L , which is not a stochastic operator and let $\alpha_i \geq 0$, $i = 1, \dots, m$, such that $\sum_{i=1}^m \alpha_i = 1$. Then it holds:*

(i) $C(S) = \{f \in L_+ : f \gg 0, Sf \leq f\}$;

(ii) $C(S)$ is a wedge;

(iii) $f_1^{\alpha_1} f_2^{\alpha_2} \dots f_m^{\alpha_m} \in C(S)$ if $f_1, f_2, \dots, f_m \in C(S)$, i.e. $C(S)$ is a logarithmic convex set.

(iv) If $f_1, f_2, \dots, f_m \in C(S)$, then

$$(9) \quad \|S(f_1^{\alpha_1} f_2^{\alpha_2} \dots f_m^{\alpha_m})\|_L \leq \|f_1^{\alpha_1} f_2^{\alpha_2} \dots f_m^{\alpha_m}\|_L \leq \|f_1\|_L^{\alpha_1} \|f_2\|_L^{\alpha_2} \dots \|f_m\|_L^{\alpha_m}$$

for any lattice seminorm on $\|\cdot\|$ on L .

(v) If $f_i, g_i \in C(S)$ for $i = 1, \dots, m$ and $\alpha \in (0, 1)$, then

$$(10) \quad \begin{aligned} \|S(f_1^\alpha g_1^{1-\alpha} + \dots + f_m^\alpha g_m^{1-\alpha})\|_L &\leq \|(f_1 + \dots + f_m)^\alpha (g_1 + \dots + g_m)^{1-\alpha}\|_L \\ &\leq \|f_1 + \dots + f_m\|_L^\alpha \cdot \|g_1 + \dots + g_m\|_L^{1-\alpha} \end{aligned}$$

for any lattice seminorm on $\|\cdot\|_L$ on L .

Proof. (i) Let $f \in L_+$ such that $f \gg 0$. We need to prove that $f \in C(S)$ if and only if $Sf \leq f$.

If $f \in C(S)$, then $f = Af \geq Sf$.

For the converse let us assume that $Sf \leq f$. Let $\varphi := f - Sf \in L_+$ and $\psi = 1 - s$, where $s(y) = \int_X s(x, y) dx$ for almost all $y \in X$. Then $\lambda := \int_X \psi(y)f(y)dy > 0$, since $\psi > 0$ on the set of positive measure and $\lambda < \infty$, since $f \in L^1(X, \mu)$. Let $A \geq S$ be a positive kernel operator on L with a kernel

$$a(x, y) = s(x, y) + \frac{1}{\lambda}\varphi(x)\psi(y)$$

for $x, y \in X$. Note that $a(x, y)$ is a $\mu \times \mu$ -measurable function on $X \times X$ such that, for all $g \in L$ and for almost all $x \in X$, $\int_X |a(x, y)g(y)| dy < \infty$, since $\psi \leq 1$, $\varphi < \infty$ almost everywhere and $g \in L^1(X, \mu)$.

For almost all $x \in X$ we have

$$(Af)(x) = \int_X s(x, y)f(y) dy + \frac{1}{\lambda}\varphi(x) \int_X \psi(y)f(y) dy = (Sf)(x) + \varphi(x) = f(x),$$

so $Af = f$. By Fubini's theorem we obtain

$$\begin{aligned} \lambda &= \int_X f(y) dy - \int_X s(y)f(y) dy = \int_X f(y) dy - \int_X \left(\int_X s(x, y)f(y) dy \right) dx \\ &= \int_X f(x) dx - \int_X (Sf)(x) dx = \int_X \varphi(x) dx. \end{aligned}$$

Therefore it follows

$$\int_X a(x, y) dx = \int_X s(x, y) dx + \frac{1}{\lambda}\psi(y) \int_X \varphi(x) dx = s(y) + \psi(y) = 1$$

for almost all $y \in X$ and so A is a stochastic operator on L . So we have proved that $C(S) = \{f \in L_+ : f \gg 0, Sf \leq f\}$. This also implies (ii).

(iii) Let $f_1, f_2, \dots, f_m \in C(S)$ and $h = f_1^{\alpha_1} f_2^{\alpha_2} \dots f_m^{\alpha_m}$. Then we have

$$Sh \leq (Sf_1)^{\alpha_1} \dots (Sf_m)^{\alpha_m} \leq f_1^{\alpha_1} \dots f_m^{\alpha_m} = h$$

by (4) and so $h \in C(S)$, which proves (iii).

The inequalities in (iv) follow from (iii) and (2), while the inequalities in (v) follow from (8), (ii), (iii) and (2). \square

Remark 2.5. If $\mu(X) < \infty$, then $L_p(X, \mu) \subset L_1(X, \mu)$ for all $p \in [1, \infty]$. In this case Theorem 2.4 can be applied to $L = L_p(X, \mu)$ equipped with e.g. standard lattice norms $\|\cdot\|_p$. Moreover, by [9, Theorem 6.6 on p.77] (see also [10]) the same applies to any rearrangement-invariant Banach function space.

Remark 2.6. If K is a strictly positive operator on L that commutes with a substochastic kernel operator S , then $f \in C(S)$ implies $Kf \in C(S)$. Indeed, $Sf \leq f$ implies $SKf = K Sf \leq Kf$.

Let \mathcal{A}_+ denote the collection of all power series

$$F(z) = \sum_{j=0}^{\infty} \alpha_j z^j$$

having nonnegative coefficients $\alpha_j \geq 0$ ($j = 0, 1, \dots$). Let R_F be the radius of convergence of $F \in \mathcal{A}_+$, that is, we have

$$\frac{1}{R_F} = \limsup_{j \rightarrow \infty} \alpha_j^{1/j}.$$

If T is an operator on a Banach space such that the spectral radius $\rho(T) < R_F$, then the operator $f(T)$ is defined by

$$f(T) = \sum_{j=0}^{\infty} \alpha_j T^j.$$

So if L is an ideal, which is also a Banach space and if $F \in \mathcal{A}_+$, then $F(S)$ commutes with S and so $f \in C(S)$ implies $F(S)f \in C(S)$, if $F(S)f \gg 0$. In particular, by choosing the exponential series and the C. Neumann series for $F \in \mathcal{A}_+$, it follows that

- (i) $f \in C(S)$ implies that $\exp(S)f \in C(S)$;
- (ii) if $f \in C(S)$ and $\lambda > \rho(S)$, then $(\lambda I - S)^{-1}f \in C(S)$.

3. NON-NEGATIVE MATRICES AND APPLICATIONS

In this final section we apply our results to (finite or infinite) non-negative matrices that define (weighted) operators on an ideal L of sequences and point out possible applications of our results.

Let R denote the set $\{1, \dots, n\}$ for some $n \in \mathbb{N}$ or the set \mathbb{N} of all natural numbers. Let $S(R)$ be the vector lattice of all complex sequences $(x_i)_{i \in R}$ and $L \subset S(R)$ an ideal. In the case when μ is the counting measure on R , then a non-negative matrix $A = [a_{ij}]_{i,j \in R}$ defines a (kernel) operator on L if $Ax \in L$ for all $x \in L$, where $(Ax)_i = \sum_{j \in R} a_{ij}x_j$.

More generally, throughout this section let $\mu(\{i\}) = \omega_i > 0$. Given ideals $L, N \subset S(R)$, a non-negative matrix A defines a (kernel) weighted operator A_ω from L to N if $A_\omega x \in N$ for all $x \in L$, where $(A_\omega x)_i = \sum_{j \in R} a_{ij}x_j\omega_j$. Note that A_ω is a positive operator, since $A_\omega x \in N_+$ for all $x \in L_+$. Recall that a positive weighted operator S_ω on L is a (column) substochastic operator, if $s_j = \sum_{i \in R} s_{ij}\omega_i \leq 1$ for all $j \in R$. The operator S_ω is stochastic, if $s(y) = 1$ for all $j \in R$, and S_ω is strictly (column) substochastic if $s(y) < 1$ for all $j \in R$. For applications of strictly substochastic operators (matrices) to the Google Page-rank model of internet usage and open Leontief model of an economy we refer the reader to [26] and the references cited there.

Similarly as in [13], [14], [22], [15] let us denote by \mathcal{L} the collection of all ideals $L \subset S(R)$ satisfying the property that $e_i = \chi_{\{i\}} \in L$. Note that $S(R) = M_0(R, \mu)$ and that for $L \in \mathcal{L}$ the set R is the carrier of L . In this setting our Theorem 2.4 says the following.

Corollary 3.1. *Let $L \subset l_\omega^1$ such that $L \in \mathcal{L}$. Assume that S_ω is a substochastic weighted operator on L , which is not stochastic, and let $\alpha_i \geq 0$, $i = 1, \dots, m$, such that $\sum_{i=1}^m \alpha_i = 1$. Then it holds:*

- (i) $C(S_\omega) = \{x \in L_+ : x \gg 0, Sx \leq x\}$;
- (ii) $C(S_\omega)$ is a wedge;
- (iii) $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m} \in C(S_\omega)$ if $x_1, x_2, \dots, x_m \in C(S_\omega)$, i.e. $C(S_\omega)$ is a logarithmic convex set.
- (iv) If $x_1, x_2, \dots, x_m \in C(S_\omega)$, then

$$\|S_\omega(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m})\|_L \leq \|x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}\|_L \leq \|x_1\|_L^{\alpha_1} \|x_2\|_L^{\alpha_2} \cdots \|x_m\|_L^{\alpha_m}$$

for any lattice seminorm on $\|\cdot\|$ on L .

- (v) If $x_i, y_i \in C(S_\omega)$ for $i = 1, \dots, m$ and $\alpha \in (0, 1)$, then

$$\begin{aligned} \|S_\omega(x_1^\alpha y_1^{1-\alpha} + \dots + x_m^\alpha y_m^{1-\alpha})\|_L &\leq \|(x_1 + \dots + x_m)^\alpha (y_1 + \dots + y_m)^{1-\alpha}\|_L \\ &\leq \|x_1 + \dots + x_m\|_L^\alpha \cdot \|y_1 + \dots + y_m\|_L^{1-\alpha} \end{aligned}$$

for any lattice seminorm on $\|\cdot\|_L$ on L .

In the following remark we discuss possible applications of our results.

Remark 3.2. (i) **Application 1.** In the well-known Arrow-Debreu model from the mathematical economy (see e.g. [4], [2], [12] and the references cited there) various commodities are exchanged, produced and consumed. The classical model deals with finitely many commodities, but as pointed out in [12], already Debreu proposed to study commodity spaces with an infinite number of commodities. Commodities may be understood as physical goods which may differ on the location or on the time that they are produced or consumed, or on the state of the world in which they become available. If we allow an infinite variation in any of this contingencies, then it is natural to consider economies with infinite number of commodities.

Let us suppose that there are countably many commodities, and that the commodity space is a positive cone L_+ of an ideal $L \subset S(R)$. A vector $x = (x_1, x_2, x_3, \dots) \in L_+$ represents a *commodity bundle*, that is, the number x_i is the amount of the i -th commodity.

If ω_i is the price for one unit of the i -th commodity, then we can introduce the price vector $\omega = (\omega_1, \omega_2, \omega_3, \dots)$. If the price vector ω is fixed for all $x \in L_+$ (as is for example in the case of state-determined prices in some EU countries for certain goods, such as gas, or prices for official or some hospital services, etc.) and $L \subset l_\omega^1(R)$, then the value of a commodity bundle x at price ω is equal to $\|x\|_{l_\omega^1} = \sum_{i \in R} x_i \omega_i$ and $\|x\|_{l_\omega^\infty} = \sup_{i \in R} x_i \omega_i = \max_{i \in R} x_i \omega_i$ equals the largest value of the commodities in x at price ω . If S_ω is a substochastic weighted operator on L , which is not stochastic, then by Corollary 3.1(iii) the wedge $C(S_\omega)$ is a logarithmic convex set. Inequalities in Propositions 2.1, 2.2 and 2.3 and Corollary 3.1(iv) and (v) provide bounds for values of considered commodity bundles. In the described setting the number $x_1^{\alpha_1}(k)x_2^{\alpha_2}(k) \cdots x_m^{\alpha_m}(k)$, for $\alpha_i > 0$, $i = 1, \dots, m$, such that $\sum_{i=1}^m \alpha_i = 1$, could represent the customers taste or *preference* for the k th commodity at m different suppliers, depending on the quality of the service eventhough the price for these services is the same at each supplier (e.g. quality of gas, quality of health services, feel-good influence on the customers, etc.). Therefore $\|x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_m^{\alpha_m}\|_{l_\omega^1}$ and $\|x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_m^{\alpha_m}\|_{l_\omega^\infty}$ represent the value of the preference vector $x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_m^{\alpha_m}$ at price ω and the largest value of the commodities in the preference vector, respectively.

(ii) **Application 2.** Let us describe what kind of role plays our Theorem 2.4(i) and (iii) (and in particular Corollary 3.1(i) and (iii)) in the theory of strategic market games (for details see [27], [6]). These results generalize results of [25, Theorem at p.1035 and Lemma at p.1036] to the infinite dimensional setting. Moreover, [25, Theorem at p.1035 and Lemma at p.1036] are merely mathematically formal versions of [27, Lemma 3 and Lemma 4 (i)]. In [27] an exchange economy with complete markets is described and a general theorem for the existence of active Nash equilibria is proved ([27, Theorem 1]). It is further shown in [27] that under replication of traders, these equilibria approach the competitive equilibria of the economy. This model was first proposed by L. Shapley and it represents one of the two possible generalisations (for the second see [6]) of the "single money" model described by Dubey and Shubik. It has a pleasant feature that it yields consistent prices.

The wedge $C(S)$ from Theorem 2.4 corresponds to the set of all possible positive multiples of price vectors that arise as trader α varies his bid in his strategic set, where a trader α is the only trader with unfixed bids. As pointed out in [27], [27, Lemma 3] "is more or less the heart of the argument" (together with the Kakutani's fixed point theorem) in the proof of the existence of Nash equilibria. Therefore, our Theorem 2.4, together with

a suitable fixed point theorem (see e.g. [8] and the references cited there for some further extensions of Kakutani's fixed point theorem), is expected to be a key ingredient in the development of the infinite dimensional version of this exchange economy.

(iii) **Applications 3 and 4.** Assume that S_ω is a substochastic weighted operator on ideal $L \subset l_\omega^1$, which is also a Banach space. By Remark 2.6(ii) $(I - S_\omega)^{-1}x \in C(S_\omega)$ if $x \in C(S_\omega)$. In a special finite dimensional case ($R = \{1, \dots, n\}$, $\omega_i = 1$ for all $i = 1, \dots, n$, $S_\omega = S$) this can be interpreted in the Google Page-rank model of internet usage and in the open Leontief model of an economy (see e.g. [26] and the references cited there)

The open Leontief model of an economy deals with the case of n industries each producing exactly one good. The production of one unit of good j requires inputs $s_{ij} \geq 0$ of the other goods i . Goods are measured in dollars-worth units, and one usually assumes that every industry runs at a profit, i.e. it costs less than a dollar to produce one dollars worth of any good. This means that the technology matrix $S = [s_{ij}]$ is strictly (column) substochastic. In order to produce a vector $p = (p_i)$ of goods, the production process consumes Sp , leaving only the excess vector $c = p - Sp$ available for outside use. One thinks of c as a "demand" vector and p as a "supply" vector, and solving for p in terms of c one gets $p = (I - S)^{-1}c$. Since S is strictly column-stochastic the spectral radius of S is less than 1, and $Y = (I - S)^{-1}$ is a nonnegative matrix given by C. Neumann's series. Y is called an impact matrix, since the ij th entry of Y is the partial derivative $y_{ij} = \frac{\partial p_i}{\partial c_j}$ and represents the increase in supply of good i in response to a 1 unit increase in the demand of good j .

The Google Page-rank model involves a discrete Markovian birth-death process that is specified by a nonnegative vector $x = (x_i)$ and a non-negative matrix $S = [s_{ij}]$. Here x_i represents the number of births (initial visits) per unit time in site i , and s_{ij} is the transition probability from site j to site i . The matrix S is strictly (column) substochastic, since there is a positive probability of death (logging off). The steady state vector p (Page-rank) satisfies $p = x + Sp$, whence we get $p = (I - S)^{-1}x$.

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Aljoša Peperko,
Faculty of Mechanical Engineering, University of Ljubljana
Aškerčeva 6, SI-1000 Ljubljana, Slovenia *and*

Institute of Mathematics, Physics and Mechanics
Jadranska 19, SI-1000 Ljubljana, Slovenia
e-mails : aljosa.peperko@fs.uni-lj.si, aljosa.peperko@fmf.uni-lj.si